Symmetric white noise can induce directed current in ratchets

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Symmetric white noise can induce directed current in periodic potentials that lack reflection symmetry (termed ratchets). The requirement for this to occur is that the white noise possesses non-Gaussian statistical properties with all its odd numbered cumulant correlation averages vanishing identically. The fluctuation-induced current is elucidated for three types of white noise: (i) symmetric white Poissonian shot noise with exponentially distributed amplitudes, (ii) two-state diffusion noise being composed of two thermal Nyquist noise sources that successively are switched *on* and *off* by dichotomic noise, and (iii) randomly flashing Gaussian white noise. Because the latter two noise sources are not composed of independent increments, the resulting ratchet dynamics $x(t)$ is non-Markovian. The current versus white-noise intensity typically exhibits a nonmonotonic dependence with a maximum assumed at a suitably tuned noise level.
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I. INTRODUCTION

In spatially periodic structures, zero-mean nonequilibrium fluctuations can induce nonzero macroscopic current (Brownian ratchets) [1]. Periodic structures are described in terms of a spatially periodic potential $V(x)=V(x+L)$ of period L. For systems with a reflection symmetry we have $V(c-x)=V(c+x)$ for some constant c. Periodic structures that lack this reflection symmetry are termed *ratchets*. Fluctuations driving the system can be symmetric or asymmetric as well. Symmetric fluctuations $\xi(t)$ are characterized by the fact that all its odd numbered cumulant averages are identically vanishing; in contrast, asymmetric noise of zero mean can possess nonvanishing odd-numbered higher-order cumulants. It is a hallmark of thermal equilibrium dynamics that (i) directed stationary motion cannot be generated by thermal fluctuations (Gaussian white noise). With a nonequilibrium thermodynamics, however, (ii) directed motion can be evoked by correlated symmetric noises in systems with a broken spatial symmetry (i.e., when the spatial potential is asymmetric) [1–5]. Furthermore, it is known that (iii) directed motion can be induced by correlated asymmetric fluctuations in reflection-symmetric systems [5] and also (iv) transport can be caused by uncorrelated (or δ-correlated) asymmetric shot noise in systems with or without a broken spatial symmetry [6]. Thus, for generation of directed transport the breaking of at least one of these symmetries is necessary. A fundamental question to be asked is what minimal statistics of noise is needed for generation of a macroscopic current. In particular, is there a possibility that a *symmetric, δ-correlated* additive noise does in fact evoke directed motion? This question will be answered with this work in the affirmative.

Let us formulate the problem in greater detail by studying the overdamped motion of Brownian particles in spatially periodic potential $V(x)$; namely, implicitly assuming a scaling that leads to dimensionless variables (see the Appendix), we consider the stochastic flow

$$\dot{x}=f(x)+\Gamma(t)+\xi(t), \tag{1}$$

where $f(x)=-dV(x)/dx$ and $\Gamma(t)$ represents thermal fluctuations that are modeled by Gaussian δ-correlated noise of zero mean,

$$\langle\Gamma(t)\rangle=0, \quad \langle\Gamma(t)\Gamma(u)\rangle=2D_T\delta(t-u), \tag{2}$$

and of strength D_T. This part models thermal Nyquist noise with its intensity being proportional to the temperature T. The process $\xi(t)$ is a ''driving force'' and models a nonequilibrum source of fluctuations.

We construct three models of symmetric and δ-correlated fluctuations $\xi(t)$, which, by virtue of statement (i), have to be nonequilibrium and non-Gaussian. The first model studied in Sec. II is symmetric Poissonian white shot noise [7]. The second noise source considered in Sec. III is composite white noise made up of two-state diffusion noise [8]: In each state the system is subject to white Gaussian noise with a given diffusion coefficient and the system randomly jumps in a dichotomic manner between these states. The third noise, the so-called randomly interrupted (or flashing) Gaussian white noise, is a limiting process of two-state diffusion noise when one of the diffusion coefficients tends to zero [9]. Then jumps between the Brownian diffusional state (a Feynman ratchet carrying zero current) and a deterministic flow (also carrying zero current) are steered by a dichotomous Markov process. It is investigated in Sec. IV. Our conclusions and a summary are presented in Sec. V.

II. SYMMETRIC POISSONIAN WHITE NOISE

Poissonian white shot noise $\xi(t)$ is defined as [7]

$$\xi(t)=\sum_{i=1}^{N(t)} z_i\delta(t-t_i), \tag{3}$$

where $N(t)$ is a Poisson counting process with a parameter λ (it is equal to a mean number of δ impulses per unit time, i.e., a mean frequency of impulses or the reciprocal of the average sojourn time between two δ kicks) and $\{z_i\}$ are weights of the δ kicks distributed according to a probability density $\rho(z)$. For symmetric white Poissonian shot noise all its odd-numbered statistical correlation cumulant averages c_{2n+1} are *zero*. This is the case when

$$\rho(z)=\rho(-z). \tag{4}$$

The first two noise correlations of $\xi(t)$ read

$$\langle \xi(t)\rangle=0, \quad \langle \xi(t)\xi(u)\rangle=2D_S\delta(t-u), \tag{5}$$

where $D_S=(1/2)\lambda\langle z_i^2\rangle$ is the shot-noise intensity. The higher-order, even-numbered cumulants $c_{2n}(t_1,t_2,\ldots,t_{2n})$ are given by [7]

$$c_{2n}(t_1,t_2,\ldots,t_{2n})=\lambda\langle z_i^{2n}\rangle\,\delta(t_1-t_2)\cdots\delta(t_{2n-1}-t_{2n}). \tag{6}$$

For $\xi(t)$ being Poissonian white noise, the output process $x(t)$ defined by Eq. (1) is a Markovian stochastic process. A master equation for the probability distribution $P(x,t)$ of it is a partial integro-differential equation of the form [10,11]

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}\,f(x)P(x,t)+D_T\frac{\partial^2}{\partial x^2}P(x,t)$$
$$+\lambda\int_{-\infty}^{\infty}[P(x-z,t)-P(x,t)]\rho(z)dz. \tag{7}$$

If one uses the relation $\exp(-z\partial/\partial x)P(x,t)=P(x-z,t)$ and the identity

$$e^{-zB}-1=-B\int_0^z e^{-sB}ds, \tag{8}$$

valid for any operator B, Eq. (7) can be recast as a continuity equation that defines the probability current $J(x,t)$. In the stationary state, when $P(x)=\lim_{t\to\infty}P(x,t)$ and $J=\lim_{t\to\infty}J(x,t)$, it takes the form

$$-D_T\frac{dP(x)}{dx}+f(x)P(x)+\lambda\int_{-\infty}^{\infty}\rho(z)\int_0^z P(x-y)dy\;dz=J. \tag{9}$$

The stationary probability current J is related to the averaged stationary velocity $\langle v\rangle$ of Brownian particles via the equality $J=\langle v\rangle/L$. Two conditions are imposed on the integro-differential equation (9): (i) the periodicity of the probability $P(x)=P(x+L)$ and (ii) the normalization of $P(x)$ over the period interval L of the ratchet potential. At this point, the above equation cannot be simplified further without specification of the density $\rho(z)$. Here we assume a two-sided exponential probabilty density, i.e.,

$$\rho(z)=(1/2A)e^{(-|z|/A)}, \quad A>0. \tag{10}$$

Then Eq. (9) can be recast as an *ordinary differential equation of third order*, i.e.,

$$\frac{D_TD_S}{\lambda}P'''(x)-\frac{D_S}{\lambda}[f(x)P(x)]''-(D_S+D_T)P'(x)$$
$$+f(x)P(x)=J, \tag{11}$$

where the prime indicates differentiation with respect to x and $D_S=\lambda A^2$.

A. Asymptotic expansions

1. High frequency of shot-noise impulses

In the limiting case $\lambda\to\infty$ with $D_S=\lambda A^2$ held fixed [which implies $A\to 0$ and $\rho(z)\to\delta(z)$] Poissonian white shot noise tends to Gaussian white noise of intensity D_S. As a consequence, the current J approaches zero. When $\lambda\gg 1$, one can expand $P(x)$ and J in a power series with respect to a small parameter λ^{-1}, i.e.,

$$P(x)=\sum_{n=0}^{\infty}\lambda^{-n}P_n(x), \quad J=\sum_{n=0}^{\infty}\lambda^{-n}J_n. \tag{12}$$

Substituting Eq. (12) into Eq. (11) and equating coefficients of equal power in λ^{-1} yield equations determining successively $P_n(x)$ and J_n. They read

$$-(D_T+D_S)P_0'(x)+f(x)P_0(x)=J_0,$$
$$-(D_T+D_S)P_n'(x)+f(x)P_n(x)=J_n+G_{n-1}(x),$$
$$n=1,2,3,\ldots, \tag{13}$$

where

$$G_n(x)=D_S[f(x)P_n(x)]''-D_TD_SP_n'''(x), \quad n=0,1,2,\ldots. \tag{14}$$

This set is supplemented with the periodicity conditions

$$P_n(x+L)=P_n(x), \quad n=0,1,2,\ldots, \tag{15}$$

and the normalization of the distribution $P(x)$ over the period L, i.e.,

$$\int_0^L P_n(x)dx=\delta_{0n}, \quad n=0,1,2,\ldots. \tag{16}$$

The problem (13)–(16) can be formally solved because Eq. (13) is a (nonhomogeneous) ordinary differential equation of first order. The zeroth-order contribution is

$$J_0=0, \quad P_0(x)=\frac{U(x)}{\int_0^L U(x)dx}, \quad U(x)=\exp\left[-\frac{V(x)}{D_T+D_S}\right]. \tag{17}$$

The higher-order contributions assume the form

$$J_n = -\frac{\int_0^L G_{n-1}(x) U^{-1}(x) dx}{\int_0^L U^{-1}(x) dx},$$

$$P_n(x) = U(x)\left[C_n - \frac{J_n}{D_T + D_S} \int_0^x U^{-1}(y) dy - \frac{1}{D_T + D_S} \int_0^x G_{n-1}(y) U^{-1}(y) dy \right],$$

$$n = 1,2,3,\ldots, \quad (18)$$

and the constants C_n are determined from Eq. (16). Both $P_n(x)$ and J_n depend on lower-order contributions via the functions $G_{n-1}(x)$ expressed by $P_{n-1}(x)$ and their derivatives; cf. Eq. (14).

The form of the first-order contribution J_1 can be simplified to a tractable form, reading

$$J_1 = -\frac{D_S^2 \int_0^L f^3(x) dx}{(D_T + D_S)^3 \int_0^L U(x) dx \int_0^L U^{-1}(x) dx}. \quad (19)$$

If the potential $V(x)$ is reflection symmetric, the integral of $f^3(x)$ over the period vanishes. On the contrary, for an asymmetric potential this integral does generally not vanish and a nonzero current can occur.

2. Low frequency of shot-noise impulses

For low frequency of impulses, when $\lambda \ll 1$, we expand $P(x)$ and J in a power series with respect to the small parameter λ,

$$P(x) = \sum_{n=0}^{\infty} \lambda^n p_n(x), \quad J = \sum_{n=0}^{\infty} \lambda^n j_n. \quad (20)$$

Equations determining $p_n(x)$ and j_n have the form

$$D_T p_0'''(x) - [f(x) p_0(x)]'' = 0,$$

$$D_T D_S p_n'''(x) - D_S [f(x) p_n(x)]'' = j_{n-1} + H_{n-1}(x),$$

$$n = 1,2,3,\ldots, \quad (21)$$

where

$$H_n(x) = (D_T + D_S) p_n'(x) - f(x) p_n(x), \quad n = 0,1,2,\ldots . \quad (22)$$

In this case it is more difficult to obtain solutions via recurrence relations as compared to the previous case (18). However, we succeeded in solving the first three equations of the system (21). The form of $p_0(x)$ follows from the first equation of Eqs. (21) and reads

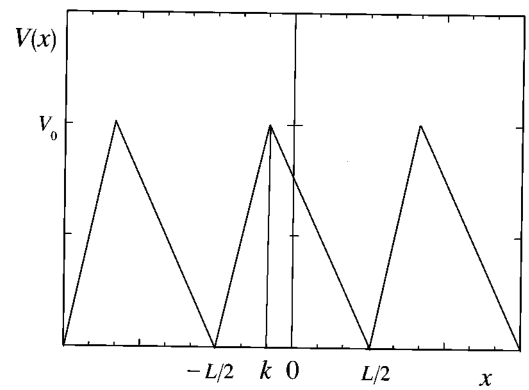

FIG. 1. Spatially periodic sawtooth potential $V(x)$ of period L, barrier height V_0, and negative asymmetry parameter k.

$$p_0(x) = \frac{W(x)}{\int_0^L W(x) dx}, \quad W(x) = \exp\left[-\frac{V(x)}{D_T} \right]. \quad (23)$$

The zeroth-order contribution j_0 follows from the second equation of Eqs. (21) for $p_1(x)$ and it turns out that $j_0 = 0$. The first-order contribution j_1 is determined from the equation for $p_2(x)$ and takes the form

$$j_1 = -\frac{1}{2} + \frac{\int_0^L W^{-1}(x) \int_0^x p_0(y) dy \; dx}{\int_0^L W^{-1}(x) dx} + \frac{1}{L}\left[\frac{\int_0^L x \; W(x) dx}{\int_0^L W(x) dx} - \frac{\int_0^L x \; W^{-1}(x) dx}{\int_0^L W^{-1}(x) dx} \right]. \quad (24)$$

The first term in the large square brackets is an equilibrium average position $\langle x \rangle$ of particles in the potential $V(x)$ (in the absence of Poissonian fluctuations), while the second term corresponds to $\langle x \rangle$, but in the inverted potential $-V(x)$. Let us note that j_1 does not depend on the shot-noise intensity D_S. From the above two asymptotic expansions, one can infer that the current J is a nonmonotonic function of λ and assumes an optimal value at some specific λ because $J \to 0$ for both $\lambda \to 0$ and $\lambda \to \infty$.

B. Sawtooth potential: Exact results

Equation (11) can be solved exactly for special forms of the potential $V(x)$. We analyze the case of a piecewise linear potential with dimensionless period $L = 2$ (Fig. 1), i.e.,

$$V(x) = \begin{cases} \frac{V_0}{L + 2k}(2x + L), & x \in [-L/2, k] \bmod L \\ \frac{-V_0}{L - 2k}(2x - L), & x \in [k, L/2] \bmod L, \end{cases} \quad (25)$$

where $V_0 > 0$ and $k \in (-L/2, L/2)$ determines the asymmetry of the potential: For $k = 0$ it is reflection symmetric; for k

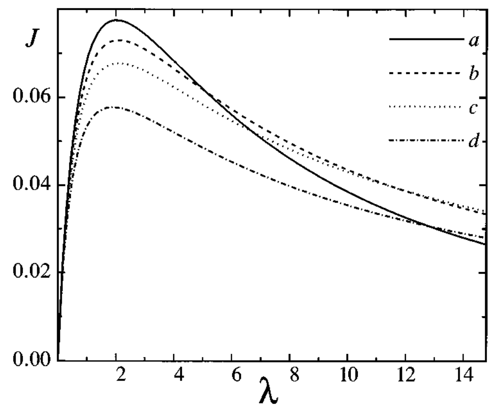

FIG. 2. Scaled, dimensionless probability current J vs switching frequency λ of δ kicks of symmetric white shot noise, in the asymmetric ($k=-0.5$) periodic sawtooth potential with period $L=2$ and barrier height $V_0=1$, for fixed thermal-noise intensity $D_T=0.1$ and selected values of white shot-noise intensity: (a) $D_S=10$, (b) $D_S=1$, (c) $D_S=0.5$, and (d) $D_S=0.1$.

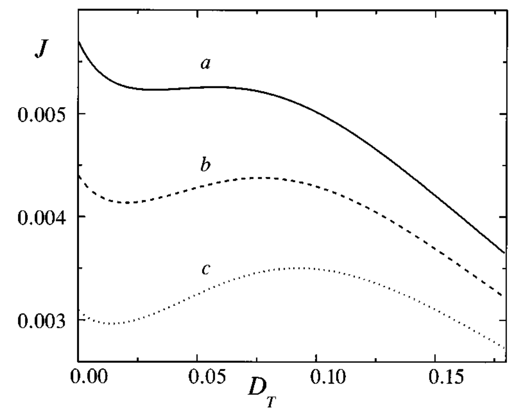

FIG. 4. Plot of the dimensionless probability current J as a function of thermal noise strength D_T (or rescaled temperature T) for fixed $\lambda=10$, $k=-0.3$, $L=2$, $V_0=1$, and selected values of shot-noise intensity: (a) $D_S=0.085$, (b) $D_S=0.078$, and (c) $D_S=0.07$.

$\neq 0$ the reflection symmetry of $V(x)$ is broken. The asymmetry is positive if $k>0$ and vice versa. If $k>0$ then the deterministic force $|f(x)|$ is smaller in the left direction or, put differently, when starting from minima of the potential $V(x)$ its slope in the x-increasing (right) direction is less than in the x-decreasing (left) direction. To calculate the current in this case, we proceed along the same way as in Sec. VI of Ref. [12]. The exact results are visualized in Figs. 2–4. For the potential (25), the asymptotics (19) and (24) read as follows: For large λ,

$$J \simeq -\frac{16kD_S^2\beta^5 e^{\beta}}{\lambda L(L^2-4k^2)^2(e^{\beta}-1)^2}, \quad \beta=\frac{V_0}{D_T+D_S}, \tag{26}$$

for small λ,

$$J \simeq -\frac{\lambda k}{L}\left[\theta e^{\theta}(e^{\theta}-1)^{-2}+\frac{1}{2}\coth(\theta/2)-\frac{2}{\theta}\right], \quad \theta=\frac{V_0}{D_T}. \tag{27}$$

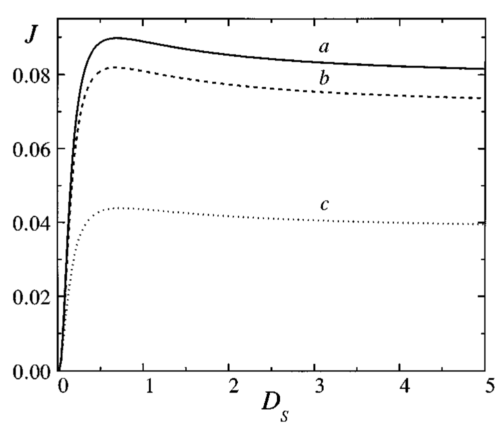

FIG. 3. Dimensionless probability current J vs white shot-noise intensity D_S for fixed $\lambda=10$, $k=-0.5$, $L=2$, $V_0=1$, and selected values of thermal-noise strength: (a) $D_T=0.002$, (b) $D_T=0.01$, and (c) $D_T=0.1$.

In contrast to Eq. (26), the current (27) does not depend on the period L of the potential itself, but only on the asymmetry parameter k/L. If the potential is symmetric (i.e., $k=0$), the current is obviously zero. For a positive asymmetry $k>0$ the current is negative; conversely, if $k<0$ the resulting current assumes positive values. This means that particles are transported into the direction opposite to motion caused by the larger force $|f(x)|$ [into the direction of steeper slope of the potential $V(x)$]. Essentially, it is the same mechanism as in flashing ratchets [1]. From time to time particles are kicked by δ impulses symmetrically to the left and to the right; between δ kicks particles move towards a neighboring minimum of $V(x)$ and this mechanism determines the direction of the net flux of particles. Details of the dependence of the current upon jumping frequency λ are depicted in Fig. 2 for several values of the shot-noise intensity. The current exhibits a bell-shaped behavior versus jump frequency λ.

Next we focus on the current versus shot-noise intensity D_S. For a wide range of the parameters λ and D_T the current grows monotonically, approaching a maximal value for an infinitely large intensity of noise. Qualitatively the same effect is observed for systems driven by asymmetric Poissonian white shot noise [6]. However, in some domain of values of the parameters, a different effect occurs (see Fig. 3): There is an optimal shot-noise intensity that maximizes the current J and approaches a nonzero value as $D_S\to\infty$. The current versus thermal noise intensity (or rescaled temperature) is depicted in Fig. 4. Like in the case of asymmetric Poissonian shot noise [12], there are two characteristic temperatures at which the current assumes locally minimal and locally (or globally) maximal values. This dependence occurs, however, for specific values of parameters. Generically, the current is a monotonically decreasing function of temperature of the system and approaches zero as $D_T\to\infty$.

III. TWO-STATE DIFFUSION NOISE

In the second model we use composite white-noise process $\xi(t)$ defined by the relation

$$\xi(t)=\frac{1}{2}[1+\eta(t)]\Gamma_1(t)+\frac{1}{2}[1-\eta(t)]\Gamma_2(t), \quad (28)$$

where $\Gamma_i(t)$ $(i=1,2)$ are independent δ-correlated Gaussian white noises of strengths D_1 and D_2, respectively. Their probabilistic characteristics are completely determined by the moments

$$\langle\Gamma_i(t)\rangle=0, \quad \langle\Gamma_i(t)\Gamma_j(u)\rangle=2\delta_{ij}D_i\delta(t-u), \quad i,j=1,2. \quad (29)$$

The process $\eta(t)=\{-1,1\}$ is a dichotomous Markovian process, which switches back and forth between two states $1\leftrightarrow-1$ with the rate ν. The process $\eta(t)$ can be expressed by a Poisson counting process $N(t)$ with a parameter ν, i.e., $\eta(t)=(-1)^{N(t)}$. It has a zero average and is exponentially correlated, i.e.,

$$\langle\eta(t)\rangle=0, \quad \langle\eta(t)\eta(u)\rangle=\exp[-|t-u|/\tau_0), \quad (30)$$

with the correlation time $\tau_0=(2\nu)^{-1}$. The composite process $\xi(t)$ is of zero mean and has a δ-correlated correlation (white noise)

$$\langle\xi(t)\rangle=0, \quad \langle\xi(t)\xi(u)\rangle=(D_1+D_2)\delta(t-u). \quad (31)$$

Its *odd-numbered* cumulants are all identically zero. Moreover, it possesses nonvanishing even-numbered δ-correlated cumulants. For example, the fourth-order cumulant has the form

$$\begin{aligned}\langle\xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4)\rangle_c\\ =(D_1-D_2)^2[e^{-2\nu|t_1-t_3|}\delta(t_1-t_2)\\ \times\delta(t_3-t_4)+e^{-2\nu|t_1-t_4|}\delta(t_1-t_3)\delta(t_2-t_4)\\ +e^{-2\nu|t_1-t_2|}\delta(t_1-t_4)\delta(t_2-t_3)], \quad (32)\end{aligned}$$

where the subscript c indicates a cumulant average.

Interestingly enough, this white noise, although being δ correlated to all orders, is non-Markovian. Note that this white noise is *not* made up of independent increments as indicated by the exponential memory contributions occuring in Eq. (32). Thus it generates a non-Markovian dynamics $x(t)$.

Two-state noise (28) is a random counterpart of deterministically modulated white noise or, put differently, of thermal fluctuations with a modulated temperature. Recently [13], ratchet systems driven by thermal noise with a periodically modulated temperature have been studied. Randomly modulated noise (28) can be realized in electrical circuits with random switching mechanisms between two different resistors. It has been observed in ultrasmall metal-oxide-semiconductor transistors in which random telegraph noise drain current fluctuations occur (fluctuations are induced by the trapping-detrapping mechanism of electrons) [14].

Next consider Eq. (1) with $\xi(t)$ substituted by composite noise in Eq. (28). Moreover, we set $\Gamma(t)\equiv0$, although this does not imply a restriction. The resulting dynamics flips between two Gaussian white-noise-driven ratchet dynamics of different intensities. In other words, the process $x(t)$ jumps with Poissonian statistics between two dynamics $\dot{x}=f(x)+\Gamma_1(t)$ and $\dot{x}=f(x)+\Gamma_2(t)$. Its probability distribution $p(x,t)=p_+(x,t)+p_-(x,t)$ is determined by the two equations [8]

$$\begin{aligned}\frac{\partial}{\partial t}p_+(x,t)=-\frac{\partial}{\partial x}f(x)p_+(x,t)+D_1\frac{\partial^2}{\partial x^2}p_+(x,t)\\ -\nu[p_+(x,t)-p_-(x,t)],\\ \frac{\partial}{\partial t}p_-(x,t)=-\frac{\partial}{\partial x}f(x)p_-(x,t)+D_2\frac{\partial^2}{\partial x^2}p_-(x,t)\\ +\nu[p_+(x,t)-p_-(x,t)], \quad (33)\end{aligned}$$

where $p_+(x,t)=p(x,\eta=+1,t)$ and $p_-(x,t)=p(x,\eta=-1,t)$. From the above equations one can construct an evolution equation for $p(x,t)$ that has the form of a continuity equation. In turn, from the continuity equation one can obtain an expression for the current $J(x,t)$. In the stationary state, J is determined by a set of two ordinary differential equations, namely,

$$-(D_1-D_2)p'_+(x)-D_2p'(x)+f(x)p(x)=J, \quad (34a)$$

$$D_1p''_+(x)-[f(x)p_+(x)]'-2\nu p_+(x)+\nu p(x)=0, \quad (34b)$$

where $p(x)$ and $p_+(x)$ are the long-time limits of $p(x,t)$ and $p_+(x,t)$, respectively.

A. Asymptotics

For large and small jump frequencies ν between two states [or, put differently, for small and large correlation times τ_0 of the dichotomic process $\eta(t)$, respectively], we expand $p(x)$, $p_+(x)$, and J in a power series with respect to ν^{-1} and ν, respectively. From Eq. (34) we then obtain

$$\begin{aligned}J\sim-\nu^{-1}\\ \times\frac{(D_1-D_2)^2\int_0^L f^3(x)dx}{(D_1+D_2)^3\int_0^L e^{2V(x)/(D_1+D_2)}dx\int_0^L e^{-2V(x)/(D_1+D_2)}dx}\\ +O(\nu^{-2}), \quad (35)\end{aligned}$$

which is valid for an arbitrary form of the ratchet potential $V(x)$. The current is identically zero for $D_1=D_2$ or/and for reflection-symmetrical potentials. Let us note that the leading-order correction is proportional to an integral of $f^3(x)$ like in Eq. (19).

For small ν, the current behaves as

$$\begin{aligned}J\sim\nu[\langle P_+(x)\rangle_1+\langle P_-(x)\rangle_2-\langle P_+(x)\rangle_2-\langle P_-(x)\rangle_1]\\ +O(\nu^2), \quad (36)\end{aligned}$$

where for any function $G(x)$

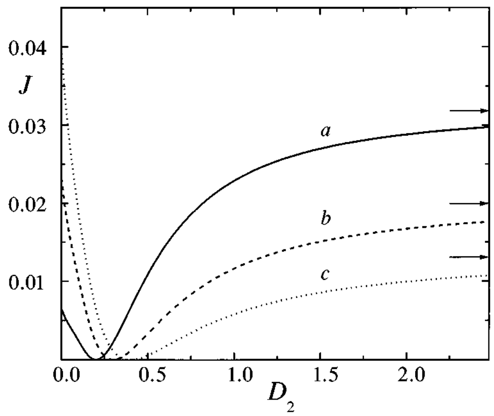

FIG. 5. Dimensionless probability current J induced by two-state diffusion noise as a function of the intensity D_2 of one of two Gaussian white noises for fixed mean switching frequency $\nu=5$ between two Gaussian noises, in an asymmetric ($k=-0.7$) periodic sawtooth potential with period $L=2$ and barrier height $V_0=1$, and three values of other Gaussian noise strength: (a) $D_1=0.2$, (b) $D_1=0.3$, and (c) $D_1=0.4$. For $D_2\to\infty$ the current saturates to a nonzero value. These saturation values for (a) $J=0.032\ldots$, and (b) $J=0.020\ldots$, and (c) $J=0.013\ldots$ have been evaluated analytically from Eq. (34) and are depicted by arrows in the figure.

$$\langle G(x)\rangle_i=\frac{\int_0^L G(x)e^{V(x)/D_i}dx}{\int_0^L e^{V(x)/D_i}dx},\quad i=1,2, \tag{37}$$

and

$$P_\pm(x)=\int_0^x p_0^\pm(y)dy. \tag{38}$$

The probability densities

$$p_0^+(x)=\frac{e^{-V(x)/D_1}}{2\int_0^L e^{-V(x)/D_1}dx},\quad p_0^-(x)=\frac{e^{-V(x)/D_2}}{2\int_0^L e^{-V(x)/D_2}dx} \tag{39}$$

are zeroth-order approximations to $p_+(x)$ and $p_-(x)$, respectively. It is remarkable that for the corresponding system driven by noise with a periodically modulated temperature of frequency ω [13], the first-order contributions are proportional to ω^{-2} for fast oscillations $\omega\gg 1$ and to ω^2 for slow oscillations $\omega\ll 1$, respectively.

B. Sawtooth potential

We consider the piecewise linear potential (25) and use the same method as in the previous case. The periodic distribution $p(x)$ is normalized to 1 over the interval $[x_0, x_0+L]$, while the periodic distribution $p_+(x)$ is normalized to 1/2 over the same interval (since it is the stationary probability Prob$\{\eta(t)=1\}=1/2$). Exact results are presented in Fig. 5.

For large ν [cf. Eq. (35)] one finds

$$J\simeq-\frac{16k(D_1-D_2)^2\beta^5e^\beta}{\nu L(L^2-4k^2)^2(e^\beta-1)^2},\quad \beta=\frac{2V_0}{D_1+D_2}. \tag{40}$$

For small ν [cf. Eq. (36)] we arrive at

$$J\simeq-\frac{\nu k}{2L}\Bigg[V_0[D_1^{-1}e^{\theta_1}(e^{\theta_1}-1)^{-2}+D_2^{-1}e^{\theta_2}(e^{\theta_2}-1)^{-2}] -\frac{D_1+D_2}{D_1-D_2}\frac{e^{\theta_2}-e^{\theta_1}}{(e^{\theta_1}-1)(e^{\theta_2}-1)}\Bigg],\quad \theta_i=V_0/D_i,\quad i=1,2. \tag{41}$$

Modulated white noise generates transport in periodic structures with a broken reflection symmetry, that is, when $k\neq 0$. The direction of current is opposite the asymmetry direction: If $k>0$ then $J<0$, and vice versa. If the two diffusion coefficients obey $D_1=D_2$, then $\xi(t)$ in Eq. (28) reduces to thermal equilibrium noise, yielding $J=0$. If one of the temperatures increases to infinity the current saturates. For example, for fixed D_1 and $D_2\to\infty$ we find from Eq. (34a) that $p(x)$ is related to $p_+(x)$ by $p(x)=p_+(x)+1/2L$. The asymptotic current then emerges as $J=L^{-1}\int_0^L f(x)p_+(x)dx$. For a sawtooth potential it can be evaluated analytically to give nonzero values, which are indicated by the three arrows in Fig. 5. The current versus the switching frequency ν exhibits a bell-shaped behavior and qualitatively is the same as in Fig. 2.

IV. RANDOMLY FLASHING GAUSSIAN WHITE NOISE

This is a limiting case of two-state diffusion noise when one of Gaussian white noises is switched off, namely, if, e.g.,

$$\xi(t)=\frac{1}{2}[1+\eta(t)]\Gamma_1(t). \tag{42}$$

Moreover, we set $\Gamma(t)=0$ for the ratchet dynamics in Eq. (1). The process (42) is white noise as well. However, the output process $x(t)$ is again non-Markovian. The dynamics of the resulting process $x(t)$ consists of two parts: the deterministic motion $\dot{x}=f(x)$ and the diffusional motion $\dot{x}=f(x)+\Gamma_1(t)$, with Poissonian statistics of jumping between them. The stationary current can be obtained from Eqs. (34) by setting $D_2=0$. For the sawtooth potential (25), the high- and low-frequency asymptotics can be obtained from Eqs. (40) and (41), carrying out the limit $D_2\to 0$. Following the previous reasoning we find that for large ν we have

$$J\simeq-\frac{16kD_1^2\beta^5e^\beta}{\nu L(L^2-4k^2)^2(e^\beta-1)^2},\quad \beta=\frac{2V_0}{D_1} \tag{43}$$

and for small ν we obtain

$$J\simeq-\frac{\nu k[1+(\theta_1-1)e^{\theta_1}]}{2L(e^{\theta_1}-1)^2},\quad \theta_1=V_0/D_1. \tag{44}$$

From these two expressions it follows that the current exhibits a bell-shaped behavior versus increasing jump frequency ν.

V. CONCLUDING REMARKS

We have shown that *symmetric* but nonthermal (non-Gaussian) *white noise* can induce directed transport in periodic structures. Three examples of such noise have been constructed: Poissonian shot noise with exponentially distributed weights of the δ kicks (other distributions of weights can be considered as well), two-state (modulated) diffusion noise, and randomly flashing Gaussian white noise. As a general property we find that the current is in the opposite direction to asymmetry k of the potential. The asymptotic current in presence of a high as well as low frequency λ of impulses (or switching rate ν between two states) has been derived. For all examples considered, we find that in the limit of strong switching rates the first-order contribution to the current involves, via the integral of $V'(x)^3$, the cubic power of the ratchet force. Such a dependence is characteristic and occurs in other ratchet problems as well [3].

In conclusion, we find that symmetric non-Gaussian white noise is sufficient to generate directed motion in periodic structures that lack reflection symmetry. The symmetric white Poissonian shot noise generates a Markovian ratchet dynamics. The fluctuation-induced current typically exhibits a bell-shaped behavior versus increasing switching frequency λ. At fixed shot-noise intensity, the current versus thermal noise intensity is generally a nonmonotonic function, approaching zero with increasing intensity D_T of Nyquist noise. For two-state diffusion noise, which is composed of two thermal Nyquist noises the ratchet dynamics $x(t)$ is non-Markovian, leading to a finite current. This situation mimics a random walker that succesively switches back and forth between two Gaussian white-noise-driven ratchet dynamics. An interesting limiting situation is obtained when one of the thermal noise sources is set to zero: The ratchet dynamics then statistically flips between a deterministic flow (carrying zero current) and an equilibrium ratchet dynamics (again carrying zero flux). Thus the resulting nonvanishing current is solely due to the switching dynamics itself. Interestingly enough, the non-Gaussian, white-noise-induced current is, as pointed out above, directed opposite to the natural direction of motion caused by the larger average force $|V'(x)|$. This feature mimics the behavior in a ''flashing'' ratchet [1,15] or, likewise, the behavior in a ''diffusion'' ratchet [1,13]. This result is in clear contrast to the characteristic feature in ratchets driven by additive colored noise (correlation ratchets [1–3]). It can be visualized by noting that the addtional source of white noise mimics a varying temperature, thus ressembling closely the physics in a diffusion ratchet: The direction of current is towards the shorter distance between the locally stable well and the neighboring barrier top. This is so because upon a ''cooling'' cycle from noise intensity $D_1 \rightarrow D_2$, where $D_1 > D_2$, the Brownian particles above and near the potential maxima escape preferably over the barrier top that is located closest to its average position.

The finding that *additive* white but non-Gaussian noise does indeed induce a finite current is not in conflict with the second law of thermodynamics: The non-Gaussian statistics generates intrinsically a stationary nonequilibrium ratchet dynamics that does not satisfy a fluctuation-dissipation relation between noise correlation and intrinsic constant friction [1]. Put differently, with the ratchet flow in Eq. (1) composed of white thermal noise and white nonthermal noise it is not possible to recast the dynamics in terms of a *single* additive Gaussian white-noise source. The fact that two-state diffusion noise and flashing Gaussian white noise can readily be experimentally realized [14] provides a good prospect that some of our results derived herein may find their way towards alternative applications of ratchet devices that are able to pump and separate mechanical or biological microparticles.

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APPENDIX

In accordance with the dissipation-fluctuation theorem, the Langevin equation for a Brownian particle of mass m reads in dimensional variables (indicated by a caret)

$$m\ddot{\hat{x}} + m\gamma\dot{\hat{x}} = -\frac{d\hat{V}(\hat{x})}{d\hat{x}} + (m\gamma k_B T)^{1/2}\hat{\Gamma}(\hat{t}) + \hat{\xi}(\hat{t}), \quad \text{(A1)}$$

where $\hat{\Gamma}(\hat{t})$ is zero-mean Gaussian white noise with the correlation function $\langle \hat{\Gamma}(\hat{t})\hat{\Gamma}(\hat{s})\rangle = 2\delta(\hat{t}-\hat{s})$ and $\hat{\xi}(\hat{t})$ is the additional nonthermal noise. The potential $\hat{V}(\hat{x})$ is spatially periodic with period L.

In the overdamped limit $m \rightarrow 0$ and $\gamma \rightarrow \infty$ in such a way that the product $m\gamma$ is fixed, Eq. (A1) reduces to the form

$$m\gamma\dot{\hat{x}} = -\frac{d\hat{V}(\hat{x})}{d\hat{x}} + (m\gamma k_B T)^{1/2}\hat{\Gamma}(\hat{t}) + \hat{\xi}(\hat{t}). \quad \text{(A2)}$$

Let us introduce next the dimensionless variables

$$t = \gamma\hat{t}, \quad x = 2\hat{x}/L. \quad \text{(A3)}$$

Then Eq. (A2) assumes the dimensionless form (1) with

$$V(x) = \frac{4\hat{V}(\hat{x})}{m\gamma^2 L^2}, \quad \xi(t) = \frac{2\hat{\xi}(\hat{t})}{m\gamma^2 L} \quad \text{(A4)}$$

and $\Gamma(t)$ is rescaled zero-mean Gaussian white noise with the correlation function $\langle \Gamma(t)\Gamma(s)\rangle = D_T\delta(t-s)$, where $D_T = 4k_B T/m\gamma^2 L^2$.

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